

Asymptotically Flat Initial Data for Gravitational Wave Spacetimes, Conformal Compactification and Conformal Symmetry

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October 11, 1998

Abstract

We study the utilization of conformal compactification within the conformal approach to solving the constraints of general relativity for asymptotically flat initial data. After a general discussion of the framework, particular attention is paid to simplifications that arise when restricting to a class of initial data which have a certain $U(1) \times U(1)$ conformal symmetry.

1 Introduction

Applying the conformal method [1] of solving the constraints of general relativity to asymptotically flat (AF) initial data, the usual choice is to pick an AF representative of the conformal equivalence class as base metric. Here instead we discuss the alternative of choosing a conformally compactified seed geometry. This formalism offers a number of advantages, e.g. it is not necessary to introduce any artificial cutoff at some finite radius in numerical work, and *all* asymptotically flat regions in multiple-black-hole initial data can be treated on an equal footing. Furthermore the method is well suited to exploit a certain conformal symmetry, which yields further simplifications.

The conformal symmetry considered here consists of a $U(1) \times U(1)$ group of *conformal isometries* associated with two commuting, orthogonal conformal Killing vector fields, and will be called (*conformal*) *toroidal symmetry*. Their action extends to a $U(1) \times U(1)$ symmetry on the many-point compactification S^3 of the physical initial hypersurface.

Based on the discussion presented here, a numerical analysis, including the existence and properties of apparent horizons, has been carried out for (i) Brill

waves (plus black holes), (ii) initial data containing a marginally outer trapped torus, and (iii) time-asymmetric initial data, where the extrinsic curvature is obtained as an exact solution for non-conformally flat geometries with conformal symmetry. These results have been discussed in [2] and [3]. The organization of this paper is as follows: first we briefly review the well known formulation of the constraints on a compactified background in Sec. 2, then we discuss toroidal symmetry and the usual Brill ansatz for axisymmetric initial data in its compactified, toroidally symmetric incarnation in Sec. 3. Section 4 discusses the constraints in the context of toroidal symmetry, in particular the case of time symmetry, while Sec. 5 deals with the case of multiple black holes.

2 Constraints and Compactification

2.1 Time Symmetry

In this section, we will briefly discuss the formulation of the constraints on a compactified background. The standard 3+1 split of Einstein's equations yields one vector (momentum) and one scalar (Hamiltonian) constraint. Choosing a maximal slice (defined as a slice of vanishing mean curvature) as the initial Cauchy surface Σ , the Hamiltonian constraint reads

$${}^3R[h_{ab}] = K_{ab}K^{ab}, \quad (1)$$

where ${}^3R[h_{ab}]$ is the scalar curvature of the spatial 3-metric h_{ab} , and K_{ab} is its extrinsic curvature. The momentum constraint on a maximal slice is

$$D_b K^b{}_a = 0. \quad (2)$$

For simplification the time-symmetric case ($K_{ab} = 0$) is discussed first. Then the momentum constraint (2) is satisfied identically, and only the Hamiltonian constraint (1) remains. In the conformal approach to solving the constraints, the physical metric is constructed from a freely specifiable seed metric \bar{h}_{ab} via conformal rescaling: $h_{ab} = \psi^4 \bar{h}_{ab}$. Assuming \bar{h}_{ab} to be asymptotically flat this ansatz transforms the Hamiltonian constraint into the linear elliptic equation

$$L_{\bar{h}}\psi \equiv \left(-\Delta_{\bar{h}} + \frac{1}{8}R_{\bar{h}}\right)\psi = 0, \quad \psi > 0, \quad \psi \rightarrow 1 \text{ at } \infty, \quad (3)$$

for the conformal factor ψ . The differential operator $L_{\bar{h}}$ is called the conformal Laplacian of the metric \bar{h} .

Note that, since the mass can be given an arbitrary value by conformal rescalings, we may in particular choose a base metric with zero mass for simplicity. The mass is then fully encoded in the conformal factor that solves the Hamiltonian constraint. For a massless metric \bar{h} , there exist regular one-point compactifications $(\bar{\Sigma}, \bar{g})$ of (Σ, \bar{h}) . So $\bar{g}_{ab} = \omega^2 \bar{h}_{ab}$ is a smooth metric on $\Sigma = \bar{\Sigma} - \Lambda$, where Λ

is the point “at infinity”, it can be extended smoothly to $(\bar{\Sigma}, \bar{g})$ and ω is a regular asymptotic distance function (RADF) of $(\bar{\Sigma}, \bar{g})$ at Λ (compare e.g. [4] and [5]). A function ω is called an RADF near a point Λ of a manifold $(\bar{\Sigma}, \bar{g})$ if it is C^∞ , positive and satisfies

$$\omega|_\Lambda = 0, \quad \bar{D}_a \omega|_\Lambda = 0, \quad (\bar{D}_a \bar{D}_b \omega - 2\bar{g}_{ab})|_\Lambda = 0, \quad \bar{D}_a \bar{D}_b \bar{D}_c \omega|_\Lambda = 0. \quad (4)$$

In asymptotically Cartesian coordinates on Σ , the asymptotic behavior of ω is given by $\omega = O(r^{-2})$ near infinity.

Going over to the compactified picture now, we consider the equation

$$\left(-\Delta_{\bar{g}} + \frac{1}{8}R_{\bar{g}}\right)G = 4\pi\delta(\Lambda), \quad (5)$$

where $\delta(\Lambda)$ is the δ -distribution centered at Λ , and \bar{g}_{ab} is the regular metric $\bar{g}_{ab} = \omega^2 \bar{h}_{ab}$ on $\bar{\Sigma}$, the one-point compactification of Σ .

The function G is required to be positive and smooth on $\bar{\Sigma} \setminus \Lambda$. Near Λ such a solution will blow up, in particular it is known [6] that

$$G = \omega^{-1/2} + \frac{m}{2} + O(\omega^{1/2}), \quad m = \text{const.}, \quad (6)$$

where ω is a RADF near Λ and the constant m is independent of the choice of RADF.

The manifold (Σ, \bar{h}) with $\bar{h}_{ab} = \omega^{-2}\bar{g}_{ab}$ is then asymptotically flat near Λ with zero ADM mass. The metric $h_{ab} = \psi^4 \bar{h}_{ab} = G^4 \bar{g}_{ab}$ however, due to Eq. (6), is asymptotic to the Schwarzschild metric of mass m :

$$\psi = \omega^{1/2}G = 1 + \frac{m}{2}\omega^{1/2} + O(\omega), \quad (7)$$

and has zero scalar curvature since $(-\Delta_{\bar{g}} + \frac{1}{8}R_{\bar{g}})G = 0$ on Σ . Its mass m is guaranteed to be positive by the positive mass theorem [7].

2.2 The General Case

We will now incorporate extrinsic curvature into the formalism. For vanishing trace of the extrinsic curvature (i.e. a maximal slice) the momentum constraint $D_a K^{ab} = 0$ is conformally invariant if K^{ab} is rescaled to $\bar{K}^{ab} = \psi^{10} K^{ab}$. This holds in particular if the conformal rescaling corresponds to compactification. Here we start with a symmetric tensor P_{ab} on the compact base manifold, which is transverse and trace-free (TT) with respect to the compactified base metric \bar{g}_{ab} . Then we obtain the TT tensor used in the AF setting by rescaling to $\bar{K}_{ab} = \omega P_{ab}$, which enters the Hamiltonian constraint in its usual formulation

$$\left(-\Delta_{\bar{h}} + \frac{1}{8}R_{\bar{h}}\right)\psi = \frac{1}{8}\bar{K}_{ab}\bar{K}^{ab}\psi^{-7}, \quad \psi \rightarrow 1 \text{ at } \infty. \quad (8)$$

The physical extrinsic curvature is then obtained by the rescaling $K_{ab} = G^{-2}P_{ab}$. In order to compute the asymptotic behavior of P_{ab} , we have to take into account both the rescaling, and the fact that the tensor components will be evaluated in different coordinate systems, which yields

$$K_{ij} = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} \frac{\partial \bar{x}^{\bar{j}}}{\partial x^j} K_{\bar{i}\bar{j}} = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} \frac{\partial \bar{x}^{\bar{j}}}{\partial x^j} P_{\bar{i}\bar{j}} G^{-2} = P_{\bar{i}\bar{j}} \times O(\omega^3), \quad (9)$$

where K_{ij} are components in an asymptotically Cartesian coordinate system x^i , and $P_{\bar{i}\bar{j}}$ are components in a coordinate system $\bar{x}^{\bar{i}}$ which is regular on S^3 near Λ . For asymptotically Schwarzschildian data ($K_{ab} = O(\omega^{3/2})$) we therefore choose $P_{ab} = O(\omega^{-3/2})$. In the more general case allowing for linear momentum we have $P_{ab} = O(\omega^{-2})$.

In order to generalize the Hamiltonian constraint to the non-time-symmetric case, we consider the equation

$$\left(-\Delta_{\bar{g}} + \frac{1}{8}R_{\bar{g}}\right)H = 4\pi\delta(\Lambda) + H^{-7}\frac{1}{8}P_{ab}P^{ab}. \quad (10)$$

This can be treated as follows: denote by G a solution to the time-symmetric Hamiltonian constraint and define

$$H = G + \phi.$$

This yields the following equation for the function ϕ

$$L_{\bar{g}}\phi = \frac{1}{8}(1 + G^{-1}\phi)^{-7}G^{-7}P_{ab}P^{ab}. \quad (11)$$

Note that this equation is regular for Schwarzschildian initial data, since then $G^{-7}P_{ab}P^{ab} = O(\omega^{1/2})$. It follows from the Appendix of [8] that there exists a unique positive solution H of Eq. (11), smooth on $\Sigma - \Lambda$, bounded and with bounded first derivatives on M .

Existence and uniqueness of solutions to the Hamiltonian constraint for given (P_{ab}, \bar{g}_{ab}) can be inferred from the time-symmetric case. It is known that a unique positive solution for (5) exists, iff the Yamabe number [9] of \bar{g}_{ab} is positive, or, equivalently if the lowest eigenvalue of $L_{\bar{g}}$ is positive. This result was effectively proven by Cantor and Brill [10]. Existence can thus be read off from the solution of the *linear* equation

$$L_{\bar{g}}\theta = \lambda\theta, \quad (12)$$

and is in particular independent of the choices of P_{ab} and Λ .

3 Toroidal Conformal Symmetry and the Brill Ansatz

In this section we will study metrics on S^3 in the context of toroidal symmetry. To start we write the standard metric on S^3 , the one-point compactification of R^3 , in toroidal coordinates (see e.g. [11]) as

$$ds^2 = \frac{d\bar{\rho}^2}{1 - \bar{\rho}^2} + (1 - \bar{\rho}^2) d\varphi^2 + \bar{\rho}^2 d\bar{z}^2, \quad (13)$$

where $0 \leq \bar{\rho} \leq 1$, $0 \leq \bar{z}, \varphi \leq 2\pi$. The vector fields $(\partial/\partial\varphi)^a$, $(\partial/\partial\bar{z})^a$ form a pair of commuting, orthogonal and hypersurface-orthogonal Killing vector fields of g spanning the surfaces of constant $\bar{\rho}$. For $\bar{\rho} \neq 0, 1$ these are flat tori, as can be seen immediately by inspection of the metric. Decompactifying to the flat metric by stereographic projection a torus $\bar{\rho} = \text{const.}$ is mapped to a standard (non-flat) torus, for $\varphi = \text{const.}$. The sets $\bar{\rho} = 0$ (respectively $\bar{\rho} = 1$) are linked great circles on S^3 , corresponding to the z -axis (respectively the circle $z = 0, r = 1$) after stereographic projection. For more details see e.g. [11], [3] or [2].

In order to achieve a convenient form of the Hamiltonian constraint for axially symmetric initial data, we use the well known Brill ansatz [12], which is usually formulated in the asymptotically flat picture, but straightforwardly translates to

$$\begin{aligned} h &= \psi^4 \left(e^{2aq(\rho, z)} (d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right) \\ &= G^4 \left(e^{2aq(\bar{\rho}, \bar{z})} \left(\frac{d\bar{\rho}^2}{1 - \bar{\rho}^2} + (1 - \bar{\rho}^2) d\bar{z}^2 \right) + \bar{\rho}^2 d\varphi^2 \right) = \psi^4 \Omega^{-2} \bar{g}. \end{aligned} \quad (14)$$

Here we have chosen the conformal factor for compactification as $\omega = \Omega = 2/(1+r^2)$, which corresponds to stereographic projection, and is compatible with the symmetries we want to exploit. Note that Ω is actually not an RADF, but 2Ω is. For the base metric \bar{g} we can now read off the ansatz

$$\bar{g} = e^{2aq(\bar{\rho}, \bar{z})} \left(\frac{d\bar{\rho}^2}{1 - \bar{\rho}^2} + (1 - \bar{\rho}^2) d\bar{z}^2 \right) + \bar{\rho}^2 d\varphi^2. \quad (15)$$

The crucial step is to choose $q = q(\bar{\rho})$. This implies that $(\partial/\partial\bar{z})^a$ is a Killing vector field (KVF) of \bar{g}_{ab} in addition to axial symmetry. At the axis $\bar{\rho} = 0$ the two conditions $q(0) = 0$, $q'(0) = 0$ are required to ensure both regularity on the axis and the correct asymptotic behavior (see [12] and [2]). A deformation of the standard metric on S^3 with $q = q(\bar{\rho})$ now respects both KVF's $(\partial/\partial\varphi)^a$ and $(\partial/\partial\bar{z})^a$. The scalar curvature simplifies to

$$\frac{R_{\bar{g}}}{8}(\bar{\rho}) = e^{-2aq} \left(\frac{3}{4} + V(\bar{\rho}) \right) \quad (16)$$

where $V(\bar{\rho}) = -\frac{a}{4} [(1 - \bar{\rho}^2) q'' - 2\bar{\rho}q']$. The action of the conformal Laplacian on axially symmetric functions is thus given by

$$L_{\bar{g}}f(\bar{\rho}, \bar{z}) = e^{-2aq(\bar{\rho})} \left(-\Delta_0 + \frac{3}{4} + V \right) f. \quad (17)$$

This simplification will be the key to turn PDEs into ODEs in the next section.

4 The Constraints in the Light of Conformal Symmetry

4.1 Existence of Solutions to The Hamiltonian Constraint

Following Sec. 2.2, to determine the existence of a solution to the Hamiltonian constraint is equivalent to positivity of the lowest eigenvalue λ_1 of $L_{\bar{g}}$. Assuming toroidal symmetry, the eigenvalue problem

$$L_{\bar{g}}\theta = \lambda\theta,$$

is separable (since $q = q(\bar{\rho})$, hence the potential depends on $\bar{\rho}$ only) and the solutions can be written as

$$\theta_{\lambda mn}(\bar{\rho}, \varphi, \bar{z}) = \vartheta_{\lambda mn}(\bar{\rho}) \sin(n\varphi + \alpha) \sin(m\bar{z} + \beta),$$

where α and β are arbitrary angles. We are thus left to solve an equation for ϑ ,

$$\left(L_{\bar{\rho}} + \frac{m^2}{1 - \bar{\rho}^2} + \frac{n^2}{\bar{\rho}^2} \right) \vartheta_{\lambda mn} = \lambda \vartheta_{\lambda mn},$$

where $L_{\bar{\rho}}$ is the *radial* part of the conformal Laplacian $L_{\bar{g}}$,

$$L_{\bar{\rho}} = -(1 - \bar{\rho}^2) \frac{\partial^2}{\partial \bar{\rho}^2} - \frac{1 - 3\bar{\rho}^2}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} + \frac{3}{4} + V(\bar{\rho}). \quad (18)$$

Since the first eigenfunction is non-degenerate [13] and has therefore to depend trivially on the angular coordinates φ and \bar{z} , we may set $m = n = 0$ when looking for the lowest eigenvalue λ_1 :

$$L_{\bar{\rho}}\vartheta_{\lambda 00} = \lambda \vartheta_{\lambda 00}. \quad (19)$$

The existence question has thus been reduced to an ODE of Sturm-Liouville type which can be solved by standard methods.

4.2 The Time-Symmetric Hamiltonian Constraint

We will now use the setup developed in Sec. (2) to convert the time-symmetric Hamiltonian constraint into a set of uncoupled ODEs. Using $q(\Lambda) = 0$, so that $\exp(2aq)\delta(\lambda) \equiv \delta(\Lambda)$, the constraint equation reads

$$\left(-\Delta_0 + \frac{3}{4} + V\right) G(\bar{\rho}, \bar{z}) = 4\pi \sqrt{2} \delta(\Lambda), \quad (20)$$

where the factor of $\sqrt{2}$ was introduced to compensate for Ω not being an ADF, so that the solution of the unperturbed ($a = 0$) case becomes $G_0 = \Omega^{-1/2}$. Nevertheless we will sloppily call G a Green function.

Before we actually transform (20) to ODEs, we will formulate it as an equation for a regular function instead of being an equation for the singular G , whose asymptotic behavior near Λ was given in (6). To remove the distributional character of this equation we write the solution as a perturbation of the Green function of the conformally flat metric:

$$G = G_0 + \phi \quad (21)$$

and get an equation for the continuous function ϕ :

$$\left(-\Delta_0 + \frac{3}{4} + V\right) \phi = -VG_0. \quad (22)$$

Depending on the choice of q the inhomogeneous term VG_0 still may show a singular behavior. If we have $q = O(\bar{\rho}^m)$ then $V = O(\bar{\rho}^{m-2})$ and

$$VG_0 = O\left(\frac{\bar{\rho}^{m-2}}{\sqrt{\bar{z}^2 + \bar{\rho}^2}}\right)$$

which is singular for $m \leq 3$. Later we will cosine transform with respect to \bar{z} and thus have to consider finiteness of the integral

$$\int VG_0 d\bar{z}$$

which will be finite for $m > 2$ ($V_0 := V(0) = 0$). The boundary condition $q = O(\bar{\rho}^2)$ translates to $m \geq 2$, so $m = 2$ is both the minimal value allowed by the boundary condition and the critical value for the finiteness of the Fourier transform of the inhomogeneous term. Note that for the critical value the divergence is logarithmic. To further regularize the equation for $m = 2$ we make the ansatz

$$G = \Omega^{-1/2} + c_1 \Omega^{1/2} + c_2 \Omega^{3/2} + \phi. \quad (23)$$

Now the constants c_1, c_2 are chosen so that the contributions of the $\Omega^{-1/2}$ and $\Omega^{1/2}$ terms cancel at Λ . The necessary choices are

$$c_1 = V_0, \quad c_2 = \frac{1}{6}V_0(2 + V_0).$$

The equation for ϕ then becomes

$$\left((1 - \bar{\rho}^2) \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1 - 3\bar{\rho}^2}{\bar{\rho}} \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1}{1 - \bar{\rho}^2} \frac{\partial^2}{\partial \bar{z}^2} - \frac{3}{4} - V \right) \phi = F(\bar{\rho}, \bar{z}), \quad (24)$$

where the function F is given by

$$F = (V - V_0) \Omega^{-1/2} + V_0 (V - V_0) \Omega^{1/2} + \frac{V_0}{6} (2 + V_0) (6 + V) \Omega^{3/2}, \quad (25)$$

which reduces to

$$F = V \Omega^{-1/2} \quad (26)$$

for $q = O(\bar{\rho}^{2+m})$.

The simple form of (24), namely that the conformal Laplacian does not explicitly depend on \bar{z} , suggests the method of expanding $\phi(\bar{\rho}, \bar{z})$ into an even Fourier (cosine) series (ϕ is invariant under reflection at the equator)

$$\phi(\bar{\rho}, \bar{z}) = \sum_{n=0}^{\infty} \phi_n(\bar{\rho}) \cos n\bar{z}. \quad (27)$$

Applying an analogous transformation to F gives a sequence of uncoupled linear ordinary differential equations for the Fourier coefficient functions ϕ_n :

$$\left((1 - \bar{\rho}^2) \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1 - 3\bar{\rho}^2}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} - \frac{n^2}{1 - \bar{\rho}^2} - \frac{3}{4} - V \right) \phi_n(\bar{\rho}) = F_n(\bar{\rho}). \quad (28)$$

Equation (28) has singular points at $\bar{\rho} = 0, 1$, which can be associated with the two axes of symmetry and the corresponding coordinate singularities. Boundary conditions are necessary to guarantee that the solutions are regular.

The axis at $\bar{\rho} = 0$ is also a physical axis of symmetry, and the boundary conditions there directly correspond to the ones in the asymptotically flat context. In addition to requiring regularity of q and the conformal factor ψ ($q, \psi \in C^2(\mathbb{R}^2)$), this implies q_ρ and ψ_ρ both have to vanish on the axis. From (23) and the definition of ψ as $\psi = \Omega^{1/2}G$ (where Ω is the compactifying conformal factor) we get

$$\psi(\bar{\rho}(x, y, z), \bar{z}(x, y, z)) = 1 + V_0 \Omega + \frac{V_0}{6} (2 + V_0) \Omega^2 + \Omega^{1/2} \phi$$

so that instead of working with ψ we impose equivalent conditions on $\phi(\bar{\rho}(x, y, z), \bar{z}(x, y, z))$, which is simply the sum of the Fourier terms $T_n = \phi_n(\bar{\rho}) \cos n\bar{z}$. For

$\bar{\rho} < 1$ the coordinate transformation from cylindrical coordinates to toroidal coordinates is regular, so we simply require $\phi_n(\bar{\rho})$ to be a regular function in this interval with $\phi' = 0$ at $\bar{\rho} = 0$.

By contrast the line $\bar{\rho} = 1$ is only an axis of symmetry for the compact manifold and the coordinate singularity there does not bear any physical meaning, and indeed the situation at $\bar{\rho} = 1$ is more delicate. The coordinate transformation from (ρ, z) to $(\bar{\rho}, \bar{z})$ is singular there. The boundary conditions on the functions $\phi_n(\bar{\rho})$ at $\bar{\rho} = 1$ have to ensure that the n th Fourier term $T_n = \phi_n(\bar{\rho}) \cos n\bar{z}$ is a regular function of the coordinates x, y, z . One can show that (see [2]) a necessary and sufficient condition for regularity of T_n is that

$$\varphi_n(\bar{\rho}) = \frac{\phi_n(\bar{\rho})}{\sqrt{1 - \bar{\rho}^{2n}}}$$

is a regular function.

We also introduce, analogously to φ_n , the functions

$$f_n = \sqrt{1 - \bar{\rho}^2}^{-n} F_n.$$

We get

$$\left((1 - \bar{\rho}^2) \frac{\partial^2}{\partial \bar{\rho}^2} + \frac{1 - (3 + 2n)\bar{\rho}^2}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} - n^2 - 2n - \frac{3}{4} - V \right) \varphi_n = f_n. \quad (29)$$

The boundary value problem of finding a positive Green function has thus been translated into the problem to find C^2 solutions to (29) for every n . An approximate solution G_N is constructed by solving for $n \leq N$, resulting in

$$G_N = \Omega^{-1/2} + V_0 \Omega^{1/2} + \frac{V_0}{6} (2 + V_0) \Omega^{3/2} + \sum_{n=0}^N \sqrt{1 - \bar{\rho}^{2n}} \varphi_n \cos n\bar{z}. \quad (30)$$

An efficient method for solving one-dimensional boundary value problems is the shooting and matching technique, which has been used in the numerical study discussed in [2].

So far we have only explicitly dealt with the axially symmetric case, by assuming that the δ -distribution source was placed on the axis $\bar{\rho} = 0$, in particular at $\bar{z} = 0$. This renders G and ϕ axially symmetric and in the Fourier series we only have to consider the $\cos n\bar{z}$ terms. The generalization of the axially symmetric to the general 3D case is straightforward. The only places where axial symmetry has been used so far is in the placing of the source, and the corresponding functional form of the Green function of the conformally flat conformal Laplacian $G_0(x, \Lambda)$, where Λ is placed on the axis $\rho = 0$, which is

$$G_0 = (1 - \sqrt{1 - \bar{\rho}^2} \cos \bar{z}).$$

For a general position of the source at $(\bar{\rho}', \bar{z}', \varphi')$ we instead have that

$$G_0 = (1 - \sqrt{1 - \bar{\rho}^2} \sqrt{1 - \bar{\rho}'^2} \cos(\bar{z} - \bar{z}') - \bar{\rho} \bar{\rho}' \cos(\varphi - \varphi')).$$

This can again be Fourier expanded in a double Fourier series in \bar{z} and φ .

Note that the structure of KVF's of the physical metric depends on the choice of “point at infinity”. The condition has been formulated as a theorem by Beig [14]: A KVF η^a of a metric g_{ab} is *also* a KVF of the decompactified metric $G^4 g_{ab}$, where G satisfies the time symmetric Hamiltonian constraint, $L_g G = 4\pi\delta\Lambda$, with Λ the point at infinity if and only if Λ is a fixed point, i.e. $\eta^a|_\Lambda = 0$.

Since the axes will turn out to be disjoint in the present case, this means that only one, or even none, of the KVF's may survive decompactification.

5 Initial Data Containing Black Holes

The formalism of solving the constraints on a conformally compactified background discussed in Sec. 2 and the simplifications due to assuming conformal symmetry described in Sec. 4 can be immediately generalized to the case when there are several asymptotic ends. We simply choose a finite number of points Λ_i ($i = 1, \dots, N$), solve numerically for the corresponding Green function G_i

$$L_{\bar{g}} G_i = 4\pi c_i \delta(\Lambda_i),$$

and define

$$h_{ab} = \left(\sum_{i=1}^N c_i G_i \right)^4 \bar{g}_{ab}, \quad (31)$$

where the “source strengths” c_i are arbitrary positive numbers. When the amplitude parameter a vanishes, these are simply the many-black-hole solutions discussed by Brill and Lindquist [15]. The existence of the G_i is guaranteed by a positive Yamabe number of \bar{g} . Thus, starting from regular initial data (with positive Yamabe number), we can always add as many black holes as we like, with arbitrary weights c_i and positions Λ_i . While the Green function yields the solution to the time-symmetric constraint, the generalization to incorporating extrinsic curvature is again straightforward along the lines of Sec. 2.2, in particular one ends up with Eq. (11). A similar approach to treat black-hole initial data, which does *not* use conformal compactification, has recently been discussed by Brandt and Brügmann [16] in the conformally flat case.

If all the Λ_i lie on the two axes, each G_i in $G = \sum_{i=1}^N c_i G_i$ corresponds to an axially symmetric (Brill-type) solution. If at least one of the sources is placed off-axis, the resulting physical geometry will not have any Killing vector fields at all.

Starting with a globally regular Brill-wave solution and adding sources, it is interesting to observe that Brill's simple positive-mass proof [12] immediately

gives positivity of mass also in this non-axially symmetric, topologically nontrivial situation: For a single source placed at Λ_1 we have that

$$G_1 = G_0 + \frac{m_1}{2} + O(G_0^{-1}),$$

where positivity of m_1 is guaranteed by Brill's argument [12]. Adding sources we get a different mass \bar{m}_1 at Λ_1 ,

$$\bar{m}_1 = 2(G - G_0)|_{\Lambda_1} = m_1 + 2 \sum_{n=2}^N G_i|_{\Lambda_1},$$

where the second term is positive by positivity of each individual G_i .

Although $\partial/\partial\bar{z}$ is *not* a KVF of the physical metric $h_{ab} = G^4\bar{g}_{ab}$, i.e. arbitrary rotations in \bar{z} are not symmetry operations, h_{ab} will possess a discrete symmetry of invariance under coordinate transformations $\bar{z} \rightarrow \bar{z} + \delta\bar{z}$ if the Λ_i are placed equally spaced at

$$\bar{z}_i = \frac{2\pi i}{N+1}, \quad \delta\bar{z} = \frac{2\pi}{N+1} \quad (32)$$

and are all equally weighted, that is $w_i = 1$ with the choice $w_0 = 1$ or more generally for any arrangement of sources that is periodic in \bar{z} .

This can be considered a generalization of inversion symmetry for two asymptotically flat sheets which corresponds to placing sources of equal strength at antipodal positions. Like any symmetry of the initial data [17] this symmetry is respected by time evolution. A nice consequence is that the symmetry carries over to minimal surfaces, that is if one finds one minimal surface, one can get another $N - 1$ minimal surfaces by shifting the coordinate \bar{z} . In particular every asymptotically flat region will show an apparent horizon. This means that for every choice of the function q the string of black holes will be surrounded by a common apparent horizon.

A particular useful formulation of toroidal symmetry is that, in the case of a general distribution of Λ_i 's, not all of the G_i 's have to be computed separately: for any Λ_1, Λ_2 lying on the same orbit of the isometry group, G_2 is given in terms of G_1 by a product of suitable rotations in φ and \bar{z} . Take e.g. $N = 2$ with Λ_1 at $(\rho = 0, \bar{z} = 0)$ and Λ_2 at $(\rho = 0, \bar{z} = \alpha)$.

In the special case $\alpha = \pi$, the physical metric h_{ab} has the mirror symmetry $\bar{z} \mapsto 2\pi - \bar{z}$. When $c_1 = c_2$, it has the inversion symmetry $\{\bar{z}\} \cup \{\bar{z} + \pi\} \mapsto \{\alpha - \bar{z}\} \cup \{\alpha - \bar{z} + \pi\}$, leaving the ‘‘throat’’ $\{\bar{z} = \alpha/2\} \cup \{\alpha/2 + \pi\}$ invariant. Thus the ‘‘throat’’ is totally geodesic with respect to h_{ab} , in particular a minimal embedded 2-sphere (a ‘‘horizon’’). When h_{ab} is conformally flat, i.e. $a = 0$, the above discrete symmetries are present irrespective of c_1, c_2 and α . Because, then, by combining a homothety with a proper conformal motion coming from the Killing vector $\frac{\partial}{\partial\bar{z}}$ on $(R^3, \bar{\delta}_{ab})$, we can always arrange for $c'_1 = c'_2, \alpha' = \pi$. The corresponding physical data is of course nothing but a time-symmetric slice of

the Kruskal spacetime with mass $m(c_1, c_2, \alpha) = 2\sqrt{2}c_1c_2(1 - \cos \alpha)^{-1/2}$. When $a \neq 0$ our solutions for $c_1 \neq c_2$ are not inversion symmetric. Placing sources of equal strength at $\rho = 0$, $\alpha_i = \frac{2\pi i}{N}$, $i = 0, 1, \dots, N-1$, we find in an analogous manner that on $R^3 \setminus \bigcup_{i=1}^n \Lambda_i$ each asymptotic end is surrounded by a throat. Thus, viewing Λ_1 as “infinity” and the other Λ_i ’s as “black holes” or “particles” we can in particular say that these objects, viewed from infinity, are so close together that, in addition to a horizon for each of them individually, there is one surrounding them all.

Acknowledgements

Much of the work presented here is based on ideas of, or was developed jointly with, my thesis advisor R. Beig, to whom I am also indebted for his support and encouragement. Thanks to C. Stanciulescu for critically reading and identifying some errors in the manuscript. Part of this work was financially supported by Fonds zur Förderung der Wissenschaftlichen Forschung, Project No. P09376-PHY.

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